Local rigidity of quasi-regular varieties

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Abstract

For a G-variety X with an open orbit, we define its boundary ∂X as the complement of the open orbit. The action sheaf S_X is the subsheaf of the tangent sheaf made of vector fields tangent to ∂X. We prove, for a large family of smooth spherical varieties, the vanishing of the cohomology groups $H^i(X, S_X)$ for $i > 0$, extending results of F. Bien and M. Brion [BB96].

We apply these results to study the local rigidity of the smooth projective varieties with Picard number one classified in [Pa08b].

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Contents

Introduction

Let X be a complex algebraic variety. Denote by T_X its tangent bundle. If X is a flag variety, it is well-known that $H^i(X, T_X) = 0$ for any $i \geq 1$ (see [De77]). By Kodaira-Spencer theory, the vanishing of $H^1(X, T_X)$ implies that X is locally rigid, *i.e.* admits no local deformation of its complex structure.

Let G be a connected reductive algebraic group over $\mathbb{C}$ and X be a smooth complete G-variety. Then X is said to be regular if it is smooth, spherical (*i.e.* has an open orbit under the action of a Borel subgroup B of G) without color (*i.e.* every irreducible B-stable divisor containing a G-orbit is G-stable). F. Bien and M. Brion proved in [BB96] that regular varieties are not locally rigid in general but they have a weaker rigidity property. Indeed, let X be a spherical variety, we denote by Ω its open G-orbit and by ∂X its complement (we shall call it the boundary of X). Denote by S_X the action sheaf of X *i.e.* the subsheaf of T_X made of vector fields tangent to ∂X. Then, combining Theorem 4.1 in [Kn94] and Proposition 2.5 in [BB96], the following result holds:

Theorem 0.1. *Let X be a complete regular variety, then $H^i(X, S_X) = 0$ for any $i > 0$.*

In this paper, we generalise this result for a larger family of spherical varieties.

Definition 0.2. Let X be a smooth complete spherical variety. Denote by n its rank, *i.e.* the minimal codimension of U-orbits in the open G-orbit of X (where U is the unipotent radical of the Borel subgroup B). The variety X is said to be quasi-regular if the following conditions hold:

(QR1) any irreducible component of the boundary ∂X is a smooth (spherical) variety of rank $n-1$;

(QR2) any G-orbit closure Y is the intersection of the irreducible components of ∂X containing Y.

Remark 0.3. A regular variety is quasi-regular (see [BB96]). Remark also that our conditions are direct generalisations of the conditions defining regular varieties in [BB96] (the fact that the boundary components intersect transversally follows from condition (QR1), see Lemma 1.6).

We will see (Lemma 1.12) that the family of quasi-regular varieties contains all smooth horospherical varieties (*i.e.* spherical varieties such that Ω is a torus bundle over a flag variety, for more details see also [Pa08a]) and all smooth spherical varieties of rank 1. The main result of the paper is the following:

Theorem 0.4. *Let X be a quasi-regular spherical variety, then $H^i(X, S_X) = 0$ for any $i > 0$.*

Apart from generalising Theorem 0.1, a motivation for Theorem 0.4 is to prove that certain spherical varieties of rank 1 are indeed locally rigid. In [Pa08b], the classification of horospherical varieties of Picard number one was achieved. More generally, in *loc. cit.*, all smooth projective two-orbits varieties with Picard number one that still have two orbits after blowing-up the closed orbit were classified. We shall say that such a variety satisfies the condition (†). The varieties satisfying (†) are spherical of rank 1. Let us describe them more precisely.

The horospherical varieties satisfying (†) have three G-orbits, the open orbit Ω and two closed orbits $Y \simeq G/P_Y$ and $Z \simeq G/P_Z$ (but they have only two orbits under the action of their automorphism group). These varieties are classified by the triples (G, P_Y, P_Z) in the following list (see [Pa08b, Th.0.1], we take the notation of [Bo75] for fundamental weights ϖ_i and $P(\varpi_i)$ is the associated maximal parabolic subgroup):

1. $(B_m, P(\varpi_{m-1}), P(\varpi_m))$ with $m \geq 3$
2. $(B_3, P(\varpi_1), P(\varpi_3))$
3. $(C_m, P(\varpi_i), P(\varpi_{i+1}))$ with $m \geq 2$ and $i \in \{1, \ldots, m-1\}$
4. $(F_4, P(\varpi_2), P(\varpi_3))$
5. $(G_2, P(\varpi_2), P(\varpi_1))$

We denote by $X^1(m)$, X^2, $X^3(m,i)$, X^4 and X^5 the corresponding varieties.

There are only two non horospherical varieties satisfying (†). We denote them by $\mathbb{X}_1$ resp. $\mathbb{X}_2$. The corresponding group G is F_4 resp. $G_2 \times \mathrm{PSL}(2)$ (and they have only two G-orbits). See [Pa08b, Definitions 2.11 and 2.12] for explicit definitions of these varieties.

The varieties $X_3(m,i)$ are the odd symplectic grassmannians studied by I. Mihai [Mi05]. They have many nice geometric properties and are in particular locally rigid. It is thus natural to ask if the other varieties satisfying (†) are also locally rigid. The answer is as follows:

Theorem 0.5. *Assume that X satisfy (†), then we have the alternative:*

- *if $X = X^5$, then $H^1(X, T_X) = \mathbb{C}$ and $H^i(X, T_X) = 0$ for any $i \geq 2$;*
- *if $X \neq X^5$, then $H^i(X, T_X) = 0$ for any $i \geq 1$.*

We shall also prove that the non trivial local deformation of the horospherical G_2-variety X^5 comes from an actual deformation to a variety homogeneous under G_2 (see Proposition 2.3).

Finally we prove (see Proposition 2.5) a characterisation of homogeneity for spherical varieties of rank 1. This simplifies some of the proofs given in [Pa08b].

1 Cohomology of the action sheaf

In this section we will prove Theorem 0.4, using Theorem 0.1. We will relate any quasi-regular variety to a regular variety by blow-ups of irreducible components of the boundary ∂X of X. We first need to recall some notation and basic facts on spherical varieties.

1.1 Spherical varieties

Let G be a reductive connected algebraic group. Let H be a closed subgroup of G. The homogeneous space G/H is said to be spherical if it has an open orbit under the action of a Borel subgroup B of G. A G/H-embedding is a normal G-variety that contains an open orbit isomorphic to G/H. Then the spherical varieties are the G/H-embeddings with G/H spherical.

Now, fix a spherical homogeneous space G/H of rank n. Then G/H-embeddings have been classified in terms of colored fans by D. Luna and T. Vust [LV83]. Let us recall a part of this theory, see [Kn91] and [Br97] for more details.

We denote by $\mathcal{D}$ the set of irreducible B-stable divisors of G/H. An element of $\mathcal{D}$ is called a color. Let M be the lattice of all characters χ of B such that there exists a non-zero element $f \in \mathbb{C}(G/H)$ such that for all $b \in B$ and $x \in G/H$ we have $f(bx) = \chi(b)f(x)$ (remark that such a f is unique up to a scalar). Denote by N the dual lattice of M and let $N_{\mathbb{Q}} = N \otimes_{\mathbb{Z}} \mathbb{Q}$. Note that N and M are lattices of rank n.

Let $D \in \mathcal{D}$, then the associated B-stable valuation $v_D : \mathbb{C}(G/H) \to \mathbb{Z}$ defines an element of $N_{\mathbb{Q}}$, denoted by $\rho(D)$. Denote by $\mathcal{V}$ the cone of $N_{\mathbb{Q}}$ generated by the image in N of the set of G-stable valuations of G/H. It is a convex polyhedral cone.

Definition 1.1. (1) A colored cone is a pair $(\mathcal{C}, \mathcal{F})$ with $\mathcal{C} \subset N_{\mathbb{Q}}$ and $\mathcal{F} \subset \mathcal{D}$ having the following properties:

- $\mathcal{C}$ is a convex cone generated by $\rho(\mathcal{F})$ and finitely many elements of $\mathcal{V}$;
- the relative interior of $\mathcal{C}$ intersects $\mathcal{V}$ non trivially;
- $\mathcal{C}$ contains no lines and $0 \notin \rho(\mathcal{F})$.

(ii) A colored face of a colored cone $(\mathcal{C}, \mathcal{F})$ is a pair $(\mathcal{C}', \mathcal{F}')$ such that $\mathcal{C}'$ is a face of $\mathcal{C}$, the relative interior of $\mathcal{C}'$ intersects non trivially $\mathcal{V}$ and $\mathcal{F}'$ is the subset of $\mathcal{F}$ of elements D satisfying $\rho(D) \in \mathcal{C}'$.

(iii) A colored fan is a finite set $\mathbb{F}$ of colored cones with the following properties:

- every colored face of a colored cone of $\mathbb{F}$ is in $\mathbb{F}$;
- for all $v \in \mathcal{V}$, there exists at most one $(\mathcal{C}, \mathcal{F}) \in \mathbb{F}$ such that v is in the relative interior of $\mathcal{C}$.

(iv) The support of a colored fan $\mathbb{F}$ is the set of elements of $\mathcal{V}$ contained in the cone of a colored cone of $\mathbb{F}$. A color of a colored cone of $\mathbb{F}$ is an element of $\mathcal{D}$ such that there exists $(\mathcal{C}, \mathcal{F}) \in \mathbb{F}$ such that $D \in \mathcal{F}$.

Theorem 1.2 (Luna-Vust)**.** *There is a bijection $X \longmapsto \mathbb{F}(X)$ between the set of isomorphism classes of G/H-embeddings and the set of colored fans. Furthermore, under this isomorphism, we have the following properties:*

- *Let X be a G/H-embedding.*

  There exists a bijection between the set of G-orbits of X and the set of colored cones of $\mathbb{F}(X)$ such that for all two G-orbits $\mathcal{O}_1$ and $\mathcal{O}_2$ in correspondence with two colored cones $(\mathcal{C}_1, \mathcal{F}_1)$ and $(\mathcal{C}_2, \mathcal{F}_2)$, we have $\mathcal{O}_1 \subset \overline{\mathcal{O}_2}$ if and only if $(\mathcal{C}_2, \mathcal{F}_2)$ is a colored face of $(\mathcal{C}_1, \mathcal{F}_1)$.

  Let $\mathcal{O}$ be a G-orbit of X associated to $(\mathcal{C}, \mathcal{F}) \in \mathbb{F}(X)$. Then for every $D \in \mathcal{F}$, we have $\overline{D} \supset \overline{\mathcal{O}}$.

  One-codimensional G-orbits correspond to one-dimensional colored cones of $\mathbb{F}(X)$ of the form $(\mathcal{C}, \emptyset)$. And G-orbits of rank $n-1$ correspond to one-dimensional colored cones of $\mathbb{F}(X)$.

  The G-orbits contained in the closure in X of a color D of X are exactly the G-orbits corresponding to colored cones of $\mathbb{F}(X)$ containing $\rho(D)$.

- *There exists a morphism between two G/H-embeddings X and X' if and only if for every colored cone $(\mathcal{C}, \mathcal{F})$ of $\mathbb{F}(X)$ there exists a colored cone $(\mathcal{C}', \mathcal{F}')$ of $\mathbb{F}(X')$ such that $\mathcal{C} \subset \mathcal{C}'$ and $\mathcal{F} \subset \mathcal{F}'$. Moreover the morphism is proper if an only if the supports of $\mathbb{F}(X)$ and $\mathbb{F}(X')$ are the same.*

We will also need the following consequence of a characterisation of Cartier divisors in spherical varieties, see [Br89, Prop.3.1]:

Proposition 1.3. *Let X be a $\mathbb{Q}$-factorial spherical variety. Then each colored cone of $\mathbb{F}(X)$ is simplicial. For any color D of $\mathbb{F}(X)$, $\rho(D)$ is in an edge of a cone of $\mathbb{F}(X)$ and if $\rho(D')$ is in the same edge, then $D = D'$ or D' is not a color of $\mathbb{F}(X)$.*

1.2 Blow-ups of quasi-regular varieties

We start with the following lemma:

Lemma 1.4. *Let X be a spherical G-variety of rank n. Then the following conditions are equivalent:*

(QR) *the irreducible components of ∂X are of rank $n-1$ and any G-orbit closure is the intersection of irreducible components of ∂X containing it;*

(VC) *For any color D of $\mathbb{F}(X)$, we have $\rho(D) \in \mathcal{V}$.*

Proof. Let us first remark, by Theorem 1.2, that the first part of condition (QR') is equivalent to say that the irreducible components $Z_0, \ldots, Z_r$ of ∂X are the closures of the G-orbits corresponding to the one dimensional colored cones of $\mathbb{F}(X)$. This implies that, any G-orbit closure that is an intersection of some Z_i's, corresponds to a colored cone of $\mathbb{F}(X)$ whose cone is generated by cones of one dimensional colored cones of $\mathbb{F}(X)$.

Suppose that there exists a color D of $\mathbb{F}(X)$ such that $\rho(D) \notin \mathcal{V}$. Let $(\mathcal{C}, \mathcal{F})$ be a colored cone of $\mathbb{F}(X)$ such that $D \in \mathcal{F}$. And let $\mathcal{O}$ the G-orbit corresponding to $(\mathcal{C}, \mathcal{F})$. Then, by the preceding remark $\overline{\mathcal{O}}$ is not the intersection of some Z_i's.

Suppose now that condition (VC) holds. Let $\mathcal{O}$ be a G-orbit and $(\mathcal{C}, \mathcal{F})$ the corresponding colored cone of $\mathbb{F}(X)$. By (VC), the cone $\mathcal{C}$ is contained in $\mathcal{V}$, so that $(\mathcal{C})$ is generated by the cones of its colored faces of dimension one. Then $Z_0, \ldots, Z_r$ corresponds to one dimensional colored cones of $\mathbb{F}(X)$, so that they are of rank $n-1$. And any G-orbit closure is the intersection of some Z_i's. $\square$

Remark 1.5. Remark that the equivalent conditions of the previous Lemma are slightly weaker than quasi-regularity, the only difference being that we do not assume the irreducible components of the boundary to be smooth.

Lemma 1.6. *($\imath$) Let X be a smooth complete spherical variety such that ∂X is the union of smooth irreducible varieties $Z_0, \ldots, Z_r$. Then the components $Z_0, \ldots, Z_r$ intersect transversally.*

($\imath\imath$) In particular if X is quasi-regular, then any closure of a G-orbit is smooth.

Proof. ($\imath$) Let us use the local structure of spherical varieties [BLV86]. Indeed, for any closed G-orbit Y of X there exists an affine open B-stable subvariety X_0 of X intersecting Y which is isomorphic to the product of the unipotent radical U_P of a parabolic subgroup P and an affine L-stable spherical subvariety V of X_0 where L is a Levi subgroup of P. Moreover V has a fixed point under L. Since X is smooth, V is also smooth. By [Lu73], any smooth affine L-stable spherical variety with a L-fixed point is an L-module. In particular V is an L-module.

Let Z_i be an irreducible component of ∂X containing Y. Let V_i be the intersection of Z_i with V. The intersection $Z_i \cap X_0$ is isomorphic to the product of U_P with V_i. In particular V_i is a proper L-stable smooth irreducible subvariety of V. By the same argument, V_i is a sub-L-module of V. Writing the decomposition of V into irreducible submodules $V = \bigoplus_{k\in K} V(\chi_k)$, there exists subsets K_i of K such that

$$V_i = \bigoplus_{k \notin K_i} V(\chi_k).$$

Let us prove that the K_i, for $i \in \{1, \dots, r\}$, are disjoint. This will imply that $V_0, \dots, V_r$, and then $Z_1, \dots, Z_r$, intersect transversally. Suppose that there exists $i \neq j$ such that $K_i \cap K_j \neq \emptyset$. Then V_i and V_j are in a same proper sub-L-module of V. It implies that Z_i and Z_j are included in a proper P-stable irreducible subvariety of X (that is not G-stable by maximality of Z_i and Z_j). Since $(G/H) \setminus (BH/H)$ is the union of the colors of G/H, we proved that Z_i and Z_j are included in the closure of a color of G/H. By Theorem 1.2, the image of this color in N must be on both edges of the colored fan corresponding to Z_i and Z_j, that is not possible.

(ii) Follows from (i) and (QR2). □

Proposition 1.7. *Let X be a quasi-regular variety. Then there exist quasi-regular varieties $X_0, \dots, X_s$ and morphisms $\phi_i : X_i \to X_{i-1}$ for all $i \in \{1, \dots, s\}$ such that $X_0 = X$, X_s is regular and ϕ_i is the blow-up of an irreducible component of ∂X_{i-1}.*

Proof. Let X be a quasi-regular variety. We proceed by induction on the number of colors of X (which is always finite). If X has no color then X is regular and there is nothing to prove. Suppose that X has a color D. Since X is smooth, by Proposition 1.3, $\rho(D)$ is in an edge of $\mathbb{F}(X)$. This implies, by Theorem 1.2, that $\overline{D}$ contains an irreducible component Z of ∂X. Let $\pi : \widetilde{X} \to X$ be the blow-up of Z in X and let E be the exceptional divisor. Because X is quasi-regular, the component Z is smooth and $\widetilde{X}$ is a smooth spherical variety. Remark that D is not a color of $\widetilde{X}$ because $\overline{D}$ does not contain E. Moreover if D' is a color of X different from D, $\rho(D')$ is in another edge of $\mathbb{F}(X)$, so $\overline{D'}$ contains a G-orbit of $X \backslash Z$ and then D' is also a color of $\widetilde{X}$.

Now let us use the description of morphism between G/H-embeddings given in Theorem 1.2, to conclude that $\mathbb{F}(\widetilde{X})$ is obtained from $\mathbb{F}(X)$ by removing the color D in all colored cones that contain this color. Indeed, since X is smooth, the colored cones of $\mathbb{F}(X)$ are simplicial. Then, if $\mathbb{F}(\widetilde{X})$ is not obtained from $\mathbb{F}(X)$ by removing some colors, there exists a one dimensional colored cone of X not in $\mathbb{F}(X)$. This gives a contradiction because ∂X and $\partial \widetilde{X}$ have the same number of irreducible components so that $\mathbb{F}(\widetilde{X})$ and $\mathbb{F}(X)$ have the same number of one dimensional colored cones.

The boundary of $\widetilde{X}$ is again the union of smooth irreducible varieties. Indeed, these components are the strict transforms of boundary components in X and the exceptional divisor. Furthermore, because of Lemma 1.4 and the fact that our blow-up only removes one color, the second condition for quasi-regularity is also satisfied. □

Remark 1.8. Note that, if X is a spherical variety, then there exists a morphism $Y \to X$ which is a sequence of blow-ups of smooth G-stable subvarieties such that Y is regular. This result follows from a general result of [RY02] on rational G-equivariant morphism (see [Br07, Proof of Corollary 4.4.2]). But in general, the smooth G-stable subvarieties that are blown-up, are not irreducible components of ∂X. For example, if Condition (VC) is not satisfied, to remove the color whose image in N is outside the valuation cone $\mathcal{V}$, we must blow-up X along a G-stable subvariety that is not an irreducible component of ∂X.

To prove Theorem 0.4, we need to study the behaviour of the action sheaf under successive blow-ups with smooth centers. We do this in the next subsection.

1.3 Action sheaf and blow-ups

Let X be a smooth variety and Y be any smooth subvariety in X. Let us denote by N_Y the normal sheaf Y in X. It this situation, it is a vector bundle on Y and there is a natural surjective morphism $T_X \to N_Y$ (here by abuse of notation we still denote by N_Y the push-forward of the normal bundle by the inclusion of Y in X). The action sheaf $T_{X,Y}$ of Y in X is the kernel of this map. In symbols, we have an exact sequence:

$$0 \to T_{X,Y} \to T_X \to N_Y \to 0.$$

If furthermore Y is of codimension 1 (i.e. a Cartier divisor), then it is not difficult to see that the action sheaf is locally free (see for example [BB96]).

Consider the blow-up $\pi : \widetilde{X} \to X$ of Y in X. Let us denote by E the exceptional divisor. In this subsection, we want to relate the action sheaf $T_{X,Y}$ with the push forward $\pi_* T_{\widetilde{X},E}$ of the action sheaf of E in $\widetilde{X}$. We shall prove the following

Lemma 1.9. *With the notations above, we have the equalities*

- $\pi_*(T_{\widetilde{X},E}) = T_{X,Y}$ *and*
- $R^i\pi_*(T_{\widetilde{X},E}) = 0$ *for all* $i > 0$.

Proof. Recall the definition of the tautological quotient bundle Q on the exceptional divisor E given by $Q = \pi^* N_Y / \mathcal{O}_E(-1)$. The following exact sequence holds (see for example [Fu98, Page 299]):

$$0 \to T_{\widetilde{X}} \to \pi^* T_X \to Q \to 0.$$

In particular, the composition of the differential of π given by $T_{\widetilde{X}} \to \pi^* T_X$ and the map $\pi^* T_X \to \pi^* N_Y$ to the normal bundle of Y factors through $\mathcal{O}_E(-1)$ the normal bundle of E. We thus have a commutative diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & T_{\widetilde{X},E} \arrow[r] & T_{\widetilde{X}} \arrow[r] \arrow[d] & \mathcal{O}_E(-1) \arrow[r] \arrow[d] & 0 \\
 & \pi^* T_{X,Y} \arrow[r] & \pi^* T_X \arrow[r] \arrow[d] & \pi^* N_Y \arrow[r] \arrow[d] & 0 \\
 & & Q \arrow[r, equal] \arrow[d] & Q \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}.$$

proving that $T_{\widetilde{X},E}$ is the kernel of the map $\pi^* T_X \to \pi^* N_Y$. Pushing forward the associated exact sequence using that $\pi_* \mathcal{O}_{\widetilde{X}} = \mathcal{O}_X$ and the fact that the pushed forward map $T_X \to N_Y$ is surjective, the result follows. $\square$

Remark 1.10. Remark that the sheaves $T_{\widetilde{X},E}$ and $\pi^*T_{X,Y}$ do not coincide on $\widetilde{X}$. Indeed, the sheaf $T_{\widetilde{X},E}$ is locally free (because E is a divisor) whereas the sheaf $\pi^*T_{X,Y}$ has a non trivial torsion part. To see this we may compute in local coordinates in an étale neigbourhood of a point in Y (or equivalently an open neigbourhood for the usual topology). In such an open subset U, the ring of X can be chosen to be $k[U] = k[x_1, \cdots, x_n]$ where the ideal of Y is $(x_1, \cdots, x_p)$. We may choose an open subset V above U such that the coordinated ring of $\widetilde{X}$ is given by $k[V] = k[y_1, \cdots, y_{p-1}, x_p, \cdots, x_n]$ and the map $V \to U$ gives rise to a morphism $k[U] \to k[V]$ given by $x_i \mapsto x_i$ for $i \geq p$ and $x_i \mapsto y_i x_p$ for $i \leq p-1$.

In these coordinates, the sheaf T_X is generated by the vectors $(\frac{\partial}{\partial x_i})_{i\in[1,n]}$ while the sheaf $T_{X,Y}$ is generated by the vectors $m_{i,j} = x_i \frac{\partial}{\partial x_j}$ for i and j in $[1,p]$ and the vectors $\frac{\partial}{\partial x_i}$ for $i > p$. These vectors satisfy the relations

$$x_k m_{i,j} = x_i m_{k,j}$$

for all i, j and k in $[1,p]$. In particular if we pull this sheaf back to V we may consider the elements $m_{i,j} - y_i m_{p,j}$ multiplying by x_p we get

$$x_p(m_{i,j} - y_i m_{p,j}) = x_p m_{i,j} - x_p y_i m_{p,j} = x_p m_{i,j} - x_i m_{p,j} = 0$$

and the element $m_{i,j} - y_i m_{p,j}$ is a torsion element.

The quotient of the sheaf $\pi^*T_{X,Y}$ by its torsion part is the image in π^*T_X of the sheaf $\pi^*T_{X,Y}$. It is also easy in these coordinates to prove that the images of $\pi^*T_{X,Y}$ and $T_{\widetilde{X},E}$ in π^*T_X coincide reproving the previous lemma. Indeed, let us first compute the image of $\pi^*T_{X,E}$ in π^*T_X. It is generated by the vectors $x_i \frac{\partial}{\partial x_j}$ for i and j in $[1,p]$ and the vectors $\frac{\partial}{\partial x_k}$ for $k > p$. Because $x_i = y_i x_p$ for $i < p$ we get the following set of generators for this image:

$$x_p \frac{\partial}{\partial x_j}, \text{ for } j \leq p \text{ and } \frac{\partial}{\partial x_k} \text{ for } k > p.$$

In particular we see that it is a locally free subsheaf of π^*T_X. To determine the image of $T_{\widetilde{X},E}$, let us first recall the map $d\pi : T_{\widetilde{X}} \to \pi^*T_X$. The sheaf $T_{\widetilde{X}}$ is locally free generated by the vectors $(\frac{\partial}{\partial y_i})_{i\in[1,p-1]}$ and $(\frac{\partial}{\partial x_k})_{k\in[p,n]}$. The differential of π is defined as follows:

$$\frac{\partial}{\partial y_i} \mapsto x_p \frac{\partial}{\partial x_i} \text{for } i < p, \ \frac{\partial}{\partial x_p} \mapsto \frac{\partial}{\partial x_p} + \sum_{i=1}^{p-1} y_i \frac{\partial}{\partial x_i} \text{ and } \frac{\partial}{\partial x_k} \mapsto \frac{\partial}{\partial x_k} \text{for } k > p.$$

The action sheaf $T_{\widetilde{X},E}$ is generated by the vectors $(\frac{\partial}{\partial y_i})_{i\in[1,p-1]}$, $x_p \frac{\partial}{\partial x_p}$ and $(\frac{\partial}{\partial x_k})_{k\in[p+1,n]}$. Its image is thus the same as the image of $\pi^*T_{X,Y}$.

More generally, we want to study the action sheaf of a subvariety Z of X containing Y as an irreducible component and such that the other components of Z meet Y transversally. More precisely, let us write the decomposition into irreducible components of Z as follows:

$$Z = \bigcup_{i=0}^{r} Z_i$$

with $Z_0 = Y$ and assume that all these components intersect transversally. We shall prove the following:

Proposition 1.11. *Let Z be a reduced subvariety of X whose irreducible components are smooth and containing Y as an irreducible component. Assume furthermore that the irreducible components of Z meet transversally. Let $\pi : \widetilde{X} \to X$ be the blowing-up of X along Y and E be the exceptional divisor. Denote by $\widetilde{Z}$ the union of E and the strict transforms of the irreducible components of Z different from Y. Then we have the equalities*

- $\pi_*(T_{\widetilde{X},\widetilde{Z}}) = T_{X,Z}$ *and*
- $R^i\pi_*(T_{\widetilde{X},\widetilde{Z}}) = 0$ *for all* $i > 0$.

Proof. To prove this result, we only need — as in the case of the previous lemma where $Z = Y$ — to prove that the images of $\pi^*T_{X,Z}$ and of $T_{\widetilde{X},\widetilde{Z}}$ in π^*T_X coincide. Indeed, we get in that case an exact sequence

$$0 \to T_{\widetilde{X},\widetilde{Z}} \to \pi^*T_X \to \pi^*N_Z \to 0$$

and the result follows by push forward. To prove the equality of these two images, we can use the local coordinate description one more time. Indeed, because Y and the other components of Z meet transversally, the equations of the other components $Z_1, \cdots, Z_r$ of Z in the coordinate ring $k[U] = k[x_1, \cdots, x_n]$ may be chosen to be $x_{p+1} = \cdots = x_{p_1} = 0$ for Z_1 and more generally $x_{p_{i-1}+1} = \cdots = x_{p_i} = 0$ for Z_i where $p = p_0 < p_1 < \cdots < p_r \leq n$. In particular the computation made in Remark 1.10 do not change and the result follows. □

Proof of Theorem 0.4. To conclude the proof of Theorem 0.4, we use Proposition 1.7 and Lemma 1.6 to prove that we are in the situation of Proposition 1.11.

We conclude by using Theorem 0.1 and the classical fact that for a morphism $f : X \to Y$ and a sheaf $\mathcal{F}$ on X, if we have $R^if_*\mathcal{F} = 0$ for $i > 0$, then $H^i(X, \mathcal{F}) = H^i(Y, f_*\mathcal{F})$. □

Now let us give examples of quasi-regular varieties. The following result implies that the result of Theorem 0.4 applies to smooth horospherical varieties and to smooth spherical varieties of rank one:

Lemma 1.12. *(ı) Smooth complete horospherical varieties are quasi-regular.*

(ıı) Smooth projective spherical varieties of rank 1 are quasi-regular (complete spherical varieties of rank 1 are all projective).

Proof. (ı) Note that any G-stable subvariety of a smooth horospherical variety is smooth [Pa06, Chap.2] then (QR1) is satisfied. Moreover, if the homogeneous space G/H is horospherical, the valuation cone is the vector space $N_{\mathbb{Q}}$ [Kn91, Cor.7.2], so that (VC) is automatically satisfied and we conclude by Lemma 1.4.

(ıı) Let X a smooth projective spherical variety of rank 1. We can assume that X is not horospherical. Then X has two orbits, so that ∂X is a flag variety. Conditions (QR1) and (QR2) are clearly satisfied. □

Remark 1.13. These results cannot be extended in the same way to spherical varieties of rank more than 1. Indeed, a G-stable subvariety of a smooth spherical variety is not necessarily smooth (see [Br94]).

2 Applications to spherical varieties of rank 1

2.1 Proof of Theorem 0.5

If G is a connected reductive algebraic group, we denote by B a Borel subgroup of G containing a maximal torus T. We denote by ϖ_i the fundamental weights of (G, B, T) with the notation of [Bo75], and by $P(\varpi_i)$ the corresponding maximal parabolic subgroup containing B. Let ρ be the sum of the fundamental weights.

Let P be a parabolic subgroup of G and V a P-module. Then the homogeneous vector bundle $G \times^P V$ over G/P is the quotient of the product $G \times V$ by the equivalence relation $\sim$ defined by

$$\forall g \in G, \forall p \in P, \forall v \in V, \quad (g, v) \sim (gp^{-1}, p.v).$$

To compute the cohomology of such vector bundles on flag varieties, we will use the Borel-Weil Theorem (see [Ak95, Chapter 4.3]):

Theorem 2.1 (Borel-Weil)**.** *Let V be an irreducible P-module of highest weight χ. Denote by $\mathcal{V}(\chi)$ the vector bundle $G \times^P V$ over G/P and by $w_0^P(\chi)$ the lowest weight of V. We have the following alternative:*

- *If there exists a root α with $\langle w_0^P(\chi) - \rho, \alpha^\vee \rangle = 0$, then, for any $i \geq 0$, $H^i(G/P, \mathcal{V}(\chi)) = 0$.*
- *Otherwise, there exists an element w of the Weyl group with $\langle w(w_0^P(\chi) - \rho), \alpha^\vee \rangle < 0$ for all positive roots α. Denote by $l(w)$ the length of w. Then we have $H^i(G/P, \mathcal{V}(\chi)) = 0$ for $i \neq l(w)$ and $H^{l(w)}(G/P, \mathcal{V}(\chi))$ is the G-module of highest weight $-w(w_0^P(\chi) - \rho) - \rho$.*

In this section we shall freely use the results of [Pa08b]. Let X be one of the horospherical varieties satisfying the condition (†) in the introduction (recall that these varieties are $X^1(m)$, X^2, $X^3(m, i)$, X^4 and X^5). Denote by P_Y and P_Z the maximal parabolic subgroups of G containing B such that the closed orbits Y and Z of X are respectively isomorphic to G/P_Y and G/P_Z. According to [Pa08b, Section 1.4], there exists a character χ of $P_Y \cap P_Z$ such that the total spaces of the normal bundles N_Y and N_Z are respectively $G \times^{P_Y} V(\chi)$ and $G \times^{P_Z} V(-\chi)$. Then, to compute the cohomology of N_Y and N_Z applying Borel-Weil Theorem, we only need to compute the lowest weights $w_0^Y(\chi)$ of the P_Y-module $V(\chi)$ and $w_0^Z(-\chi)$ of the P_Z-module $V(-\chi)$. Remark that, if P is a parabolic subgroup of G, w_0^P is the longest element of the Weyl group fixing the characters of P. We summarize in the following table, the value of these data in each case.

X	Type of G	P_Y	P_Z	χ	$w_0^Y(\chi)$	$w_0^Z(-\chi)$
$X^1(m)$	B_m	$P(\varpi_{m-1})$	$P(\varpi_m)$	$\varpi_m - \varpi_{m-1}$	$-\varpi_m$	$-\varpi_1 + \varpi_m$
X^2	B_3	$P(\varpi_1)$	$P(\varpi_3)$	$\varpi_3 - \varpi_1$	$-\varpi_3$	$-\varpi_2 + \varpi_3$
$X^3(m, i)$	C_m	$P(\varpi_i)$	$P(\varpi_{i+1})$	$\varpi_{i+1} - \varpi_i$	$-\varpi_{i+1} + \varpi_i$	$-\varpi_1$
X^4	F_4	$P(\varpi_2)$	$P(\varpi_3)$	$\varpi_3 - \varpi_2$	$-\varpi_4$	$-\varpi_1 + \varpi_3$
X^5	G_2	$P(\varpi_1)$	$P(\varpi_2)$	$\varpi_2 - \varpi_1$	$-\varpi_2 + 2\varpi_1$	$-\varpi_1$

In each case, by the Borel-Weil Theorem, one of the normal bundle has cohomology in degree 0 (and only in degree 0). This was already used in [Pa08b]. And the other normal bundle has no cohomology except in the last case where N_Y has cohomology in degree 1 (and only in degree 1). Indeed, when $X = X^5$, we have $H^1(X, N_Y) = \mathbb{C}$.

When X is one of the two spherical varieties $\mathbb{X}_1$ and $\mathbb{X}_2$, denote by P_Y the parabolic subgroup of G containing B such that the unique closed orbit Y is isomorphic to G/P_Y. Let χ be the character of P_Y such that the total space of the normal bundle N_Y is $G \times^{P_Y} V(\chi)$ and let $w_0^Y(\chi)$ be the lowest weight of the P_Y-module $V(\chi)$. Then we have the following table.

X	G	P_Y	χ	$w_0^Y(\chi)$
$\mathbb{X}_1$	F_4	$P(\varpi_3)$	$\varpi_1 - \varpi_3$	$-\varpi_2 + \varpi_3$
$\mathbb{X}_2$	$G_2 \times \mathrm{PSL}(2)$	$P(\varpi_1) \cap P(\varpi_0)$	$\varpi_2 - 2\varpi_1 - 2\varpi_0$	$-\varpi_2 + \varpi_1 - 2\varpi_0$

In both cases, the Borel-Weil Theorem implies the vanishing of the cohomology of the normal bundle of Y in X.

Then Theorem 0.5 is a corollary of Theorem 0.4 using the long exact sequence defining the action sheaf:

$$0 \longrightarrow S_X \longrightarrow T_X \longrightarrow N_{\partial X/X} \longrightarrow 0$$

where $\partial X = Y \cup Z$ if X is horospherical and Y in the other cases.

Remark 2.2. This result gives an example of an horospherical variety such that $H^1(X, T_{X,Y}) \neq 0$ with Y is a G-stable subvariety of X (different from ∂X). Indeed, let X be the smooth projective non-homogeneous horospherical G_2-variety of Picard number one (*i.e.* the variety X^5). We have the following short exact sequence

$$0 \longrightarrow S_X \longrightarrow T_{X,Y} \longrightarrow N_Z \longrightarrow 0.$$

This implies the equalities $H^1(X, T_{X,Y}) = H^1(X, N_Z) = \mathbb{C}$.

2.2 Explicit deformation of the horospherical G_2-variety X^5

In the unique case where X is not locally rigid, we prove that the local deformation comes from a global deformation of X. We describe explicitly the unique deformation of X in the following proposition.

Proposition 2.3. *The variety X^5 has a deformation to the orthogonal grassmannian $\mathrm{Gr}_q(2, 7)$.*

Proof. Let us denote by $(z_0, z_1, z_2, z_3, z_{-1}, z_{-2}, z_{-3})$ the basis of the imaginary octonions $\mathrm{Im}(\mathbb{O})$ as in [Pa08b, Section.2.3]. Then we have

$$X^5 = \overline{G_2.[z_3 + z_1 \wedge z_3]} \subset \mathbb{P}(V(\varpi_1) \oplus V(\varpi_2)).$$

Now for all $t \in \mathbb{C}^*$, define $x_t := z_3 + t(z_0 \wedge z_3 + z_1 \wedge z_2) + z_1 \wedge z_3 \in V(\varpi_1) \oplus V(\varpi_2)$ and set

$$X_t^5 := \overline{G_2.[x_t]} \subset \mathbb{P}(V(\varpi_1) \oplus V(\varpi_2)).$$

The limit of the varieties X_t when t goes to 0 is X^5. We conclude the proof thanks to the following lemma. ☐

Lemma 2.4. *For all $t \in \mathbb{C}^*$, X_t is isomorphic to the orthogonal grassmannian* $\mathrm{Gr}_q(2,7)$.

Proof. For τ in $\mathbb{C}$, let $\phi_\tau \in G_2$ be the automorphism of the $\mathbb{O}$ sending $(1, z_0, z_1, z_2, z_3, z_{-1}, z_{-2}, z_{-3})$ to $(1, z_0 + \tau z_1, z_1, z_2 + \frac{\tau}{2} z_3, z_3, z_{-1} + \tau z_0 + \frac{\tau^2}{2} z_1, z_{-2}, z_{-3} - \tau z_{-2})$. Then we have that the element $\phi_{-2/3t}(x_t) = z_3 + t(z_0 \wedge z_3 + z_1 \wedge z_2)$ is in the open orbit of X_t. This proves the lemma, by [Pa08b, Prop 2.34]. □

2.3 Automorphisms

In [Pa08b], a case by case analysis was necessary to prove that certain spherical varieties of rank 1 are homogeneous. The following result gives a more uniform way to check the homogeneity of these varieties.

Proposition 2.5. *Let X be a smooth projective spherical variety of rank 1. Then* $\mathrm{Aut}(X)$ *acts transitively on X if and only if for all closed G-orbit Y of X, the normal bundle $N_{Y/X}$ has a non-zero section.*

Proof. Let X be a smooth spherical variety of rank 1. Recall that, in that case, ∂X is one closed orbit or the union of two-closed orbits.

Suppose that there exists a closed G-orbit Y of X such that $H^0(X, N_Y) = 0$ then we have the equality $H^0(X, T_{X,Y}) = H^0(X, T_X)$. Thus the identity connected component of $\mathrm{Aut}(X)$ stabilises Y and then $\mathrm{Aut}(X)$ cannot act transitively on X.

Suppose now that, for all closed G-orbit Y of X, the normal bundle N_Y has a non-zero section. This implies, by the Borel-Weil Theorem, that for all closed G-orbit Y of X, we have $H^1(X, N_Y) = 0$. We thus have the following exact sequence

$$0 \longrightarrow H^0(X, T_{X,Y}) \longrightarrow H^0(X, T_X) \longrightarrow H^0(Y, N_Y) \longrightarrow 0.$$

Indeed, if X has only one closed G-orbit, then $H^1(X, T_{X,Y}) = 0$ because $T_{X,Y} = S_X$. If X has two closed G-orbits Y and Z, then we have the following exact sequence

$$0 \to S_X \to T_{X,Y} \to N_Z \to 0$$

and we obtain $H^1(X, T_{X,Y}) = 0$ from Theorem 0.4. This implies that Y cannot be stabilised by $\mathrm{Aut}(X)$ and the result follows. □

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